SYNCHRONISATION OF ALMOST ALL TRAJECTORIES OF A RANDOM DYNAMICAL SYSTEM

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Abstract. It has been shown by Le Jan that, given a memoryless-noise random dynamical system together with an ergodic distribution for the associated Markov transition probabilities, if the support of the ergodic distribution admits locally asymptotically stable trajectories, then there is a random attracting set consisting of finitely many points, whose basin of forward-time attraction includes a random full measure open set. In this paper, we present necessary and sufficient conditions for this attracting set to be a singleton. Our result does not require the state space to be compact, but holds on general Lusin metric spaces (in both discrete and continuous time).

1. Introduction

We consider a random dynamical system (RDS) φ on a metric space X (which is taken to be a Borel subset of a separable complete metric space), driven by memoryless stationary noise, in either discrete or continuous time. (For a rigorous formulation of this, see Section 2.1.) Since the noise is stationary and memoryless, the trajectories of φ are homogeneous Markov processes. Given an ergodic distribution ρ for the associated Markov transition probabilities, we say that φ is *stable with respect to* ρ if there is a positive-measure set of noise realisations under which some trajectories in the support of ρ are locally asymptotically stable.[1] When X is a manifold, this property is typically implied by negativity of the Lyapunov spectrum.

It has been shown in [14] that if φ is stable with respect to ρ then there exists $n \in \mathbb{N}$ with the property that for almost every noise realisation, ρ-almost all of the state space X can be partitioned into n open (noise-dependent) regions of equal ρ-measure, such that trajectories starting in the same region synchronise as time tends to ∞ but trajectories starting in different regions do not synchronise as time tends to ∞.

In this paper, we will give necessary and sufficient conditions for the number of regions n to be 1; this situation is precisely the situation that for almost every noise realisation, the trajectories of ρ-almost all initial conditions in X are locally asymptotically stable and synchronise with each other. We describe this scenario by saying that φ *is* ρ*-almost everywhere stably synchronising*.

Let us now describe our result in more detail. It is straightforward to show that if φ is ρ-almost everywhere stably synchronising then there is a ρ-full set $A \subset X$ such

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[1] It follows from this that for almost every realisation of the noise, the trajectory of ρ-almost every initial condition in X is asymptotically stable.

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that, given any initial conditions $x, y \in A$ and any open $U \subset X$ with $\rho(U) > 0$, for almost every noise realisation the trajectories of x and y will (at some point in time) simultaneously be in U. Assuming that φ is stable with respect to ρ, we will show that the converse also holds (that is to say, the existence of such a ρ-full set $A \subset X$ implies that φ is ρ-almost everywhere stably synchronising). Of course, verifying the existence of such a set A is difficult; but we will show that it is sufficient to verify much less than this—namely:

Theorem. *Assume φ is stable with respect to ρ. Suppose there exist a ρ-positive measure set $A_1 \subset X$ and a ρ-full set $A_2 \subset X$ such that for all $(x, y) \in A_1 \times A_2$ there is a ρ-transitive point $p(x, y)$ so that, given any neighbourhood $U \subset X$ of $p(x, y)$, with strictly positive probability the trajectories of x and y will (at some point in time) simultaneously be in U. Then φ is ρ-almost everywhere stably synchronising.*

Here, a *ρ-transitive point* is an initial condition in the support of ρ from which every ρ-positive measure open set is accessible. Since ρ is ergodic, ρ-almost every initial condition is ρ-transitive.

The proof of our result is based on a generalisation of a method in [10].

To illustrate our result, we will show that the "double-well potential with additive noise" as considered in [7] exhibits almost sure forward-time synchronisation of the trajectories of any given pair of initial conditions.[2]

Let us now give a brief introduction to synchronisation of trajectories in random dynamical systems, and an overview of existing results on the topic.

Synchronisation of trajectories is manifested physically in the important phenomenon of "noise-induced synchronisation", where two or more non-interacting processes starting at different initial states are caused to synchronise with each other due to simultaneous exposure to the same source of external random perturbations. This phenomenon was initially described in the early 1980s ([20], [1]), and since then, there have been numerous case studies of noise-induced synchronisation (analytical, numerical and experimental); see e.g. [22] and references therein. The theory of random dynamical systems is central in the mathematical study of noise-induced synchronisation, since the evolutions of the processes affected by the random perturbations can typically be regarded as simultaneous trajectories of one RDS under the same noise realisation.

In analytical studies of synchronisation of trajectories in RDS, a key concept that is often considered is *Lyapunov exponents*. Lyapunov exponents are primarily suited to the context of spatially smooth RDS on Euclidean space or on a manifold, and they measure "infinitesimal-scale repulsion of trajectories". When the *maximal Lyapunov exponent* associated to a trajectory exists and is negative, it typically follows that the trajectory is locally asymptotically stable. Given an ergodic distribution ρ for the Markov transition probabilities of the RDS, provided some weak conditions are met, there will exist a value $\lambda_\rho \in \mathbb{R} \cup \{-\infty\}$ such that for almost every noise-realisation, for ρ-almost every initial condition the maximal Lyapunov exponent associated to the corresponding trajectory exists and is equal to λ_ρ. (See e.g. the start of Section 2 of [14].) We refer to λ_ρ as the *maximal Lyapunov exponent associated to ρ*.

[2]The results in [7] yield that the noisy double-well potential exhibits global synchronisation in a *pullback* sense. (Nonetheless, combining this with forward-time local asymptotic stability does provide an alternative way to obtain almost sure forward-time synchronisation of the trajectories of any two given initial conditions.)

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As we have said, negativity of the maximal Lyapunov exponent typically implies *local* asymptotic stability of trajectories; the natural question is then to find conditions under which we can deduce some “larger-scale” synchronisation of trajectories. We will now mention some existing results pertaining to this question.

In [19], necessary and sufficient conditions are found for a memoryless-noise RDS on a *compact* space to exhibit almost sure synchronisation of the trajectories of any given pair of initial conditions, together with almost sure local asymptotic stability of the trajectory of any given initial condition. One of the key differences between the result of [19] and the result of our present paper is that, in our present paper, compactness of the state space is not needed. However, it is worth saying that when the state space *is* compact, the necessary and sufficient conditions for “stable synchronisation” given in [19] also serve as *sufficient* conditions for ρ-almost-everywhere stable synchronisation; and when these conditions are satisfied, they are likely to be easier to verify than the necessary and sufficient conditions given in this present paper for ρ-almost-everywhere stable synchronisation.

In [10], discrete-time diffeomorphic RDS on a compact manifold are considered. Theorems 1.1[3] and 1.2 of [10] provide sufficient conditions for almost sure synchronisation of the trajectories of any given pair of initial conditions, in either the whole manifold or a suitable open subset thereof. Theorems 1.1 and 1.2 of [10] can in fact be derived as particular cases of the main result in [19]. Nonetheless, the basic idea of the proof of [10, Theorem 1.1] can be generalised well beyond the context of a diffeomorphic RDS on a compact manifold. Specifically, the basic idea of the proof is that, given any set S of initial conditions, if the subsequent trajectories are able to simultaneously reach an arbitrarily small neighbourhood of some point p, and if the trajectory starting at p is itself able to reach an open region U within which it is possible for all trajectories to synchronise, then it is possible that the trajectories of all the initial conditions in S will eventually enter U and then synchronise. It is precisely by combining this idea with [14, Proposition 2] that the main result of our present paper has been obtained.

In [7], sufficient conditions are given for a RDS on a separable complete metric space to exhibit mutual synchronisation “in probability” of the trajectories of all initial conditions.[4] As an application, large classes of ordinary differential equations in Euclidean space are shown to exhibit such synchronisation when Gaussian white noise is added to the right-hand side. In [16], a different set of conditions for “weak synchronisation” as defined in [7] is given from that given in [7]. The conditions in [16] turn out to be necessary and sufficient for the type of system under consideration (namely, memoryless-noise RDS admitting a stationary distribution). These conditions are shown to be readily verifiable in the context of isotropic stochastic flows with non-positive maximal Lyapunov exponent; in particular, the case where the maximal Lyapunov exponent is zero cannot be treated by the methods in [7]. The key differences between the versions of synchronisation considered in [7, 16] and the version of synchronisation considered in this present paper are: firstly, in the present paper, the synchronisation is sample-pathwise in forward time (as opposed to synchronisation defined in terms of convergence in

[3]In the statement of [10, Theorem 1.1], it seems that the required additional assumption that m is the only stationary probability measure is missing.

[4]More specifically, the phenomena considered in [7] are the existence of a weak global point-attractor and the existence of a weak global attractor.

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probability); and secondly, in this present paper, the synchronisation has to apply across almost all initial conditions but not necessarily across all initial conditions. (Hence in particular, our conditions do not rule out the existence of multiple different ergodic measures for the Markov transition probabilities.)

In [3], Wiener-driven stochastic differential equations on a compact manifold are considered. Certain non-degeneracy conditions on the vector fields are assumed, implying in particular that there is a unique ergodic distribution ρ for the Markov transition probabilities, and that ρ is equivalent to the Riemannian measure (under any Riemannian metric). One of the results proved (Theorem 4.10) is that if the maximal Lyapunov exponent λ_ρ associated to ρ is negative and any two distinct trajectories are always able to come closer together, then the system exhibits almost sure synchronisation of the trajectories of any given pair of initial conditions. This result is, in fact, a special case of the main result of [19]. However, remarkably, if we replace the condition that λ_ρ is negative with the condition that $\lambda_\rho = 0$, a further result of [3] (Corollary 5.12) gives that the system will still exhibit a kind of global-scale synchronisation (where the notion of synchronisation involved is based on convergence in probability).

Now there also exist several results to the effect that if a RDS has some order-preserving or orientation-preserving property, then under some weak conditions synchronisation is guaranteed: see e.g. [6] for order-preserving RDS on $\mathbb{R}$, [8] for order-preserving RDS on more general partially ordered spaces, and [15] and [11] for orientation-preserving RDS on a circle.

The structure of the paper will be as follows: In Section 2, we will present the formal setup, introduce some key definitions and results, and then state our main result (Theorem 2.11). We will also present the double-well potential example. In Section 3, we give the proof of our main result, first introducing some preliminary theory of RDS as necessary.

2. The basic setup and our result

Throughout this paper, given measurable spaces $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ and a measurable map $f \colon X \to Y$, for any probability measure μ on $(X, \mathcal{X})$ we write $f_*\mu$ for the corresponding probability measure $\mu(f^{-1}(\,\cdot\,))$ on $(Y, \mathcal{Y})$.

2.1. **The setup: RDS with memoryless noise.** A "random dynamical system with memoryless noise" consists of two components: a "memoryless" filtered measure-preserving flow, representing the "noise"; and an adapted cocycle over this flow acting on the state space.

Let $\mathbb{T}$ be either $\mathbb{Z}$ or $\mathbb{R}$, and let $\mathbb{T}^+ := \mathbb{T} \cap [0, \infty)$. Let $\bar{\mathbb{T}} := \mathbb{T} \cup \{-\infty, \infty\}$, and let $\bar{\mathbb{T}}^+ := \mathbb{T}^+ \cup \{\infty\}$. Let $(\Omega, \mathcal{F})$ be a measurable space, and let $(\mathcal{F}_s^{s+t})_{s \in \mathbb{T},\, t \in \mathbb{T}^+}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that

(i) $\mathcal{F}_{t_1}^{t_2} \subset \mathcal{F}_{t_0}^{t_3}$ for all $t_0 \leq t_1 \leq t_2 \leq t_3$ in $\mathbb{T}$;
(ii) $\sigma(\mathcal{F}_s^{s+t} : s \in \mathbb{T}, t \in \mathbb{T}^+) = \mathcal{F}$.

We will use the following notations:

$$\begin{aligned}
\mathcal{F}_s^\infty &:= \sigma(\mathcal{F}_s^{s+t} : t \in \mathbb{T}^+) \quad \text{for any } s \in \mathbb{T} \\
\mathcal{F}_\infty^\infty &:= \bigcap_{s \in \mathbb{T}} \mathcal{F}_s^\infty \\
\mathcal{F}_{-\infty}^t &:= \sigma(\mathcal{F}_{t-s}^t : s \in \mathbb{T}^+) \quad \text{for any } t \in \mathbb{T} \\
\mathcal{F}_{-\infty}^{-\infty} &:= \bigcap_{t \in \mathbb{T}} \mathcal{F}_{-\infty}^t
\end{aligned}$$

It will also be useful to have the convention that $\mathcal{F}_{-\infty}^\infty := \mathcal{F}$. Let $(\theta^t)_{t\in\mathbb{T}}$ be a group of $(\mathcal{F},\mathcal{F})$-measurable functions $\theta^t \colon \Omega \to \Omega$ such that $\theta^\tau \mathcal{F}_s^t = \mathcal{F}_{s-\tau}^{t-\tau}$ for all $s, t, \tau \in \mathbb{T}$ with $s \leq t$. Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$ with the following properties:

(i) $\theta_*^t \mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{T}$;

(ii) for each $t \in \mathbb{T}$, $\mathcal{F}_{-\infty}^t$ and $\mathcal{F}_t^\infty$ are independent σ-algebras under $\mathbb{P}$.

Property (i) represents stationarity of the noise, and property (ii) represents memorylessness of the noise. As in [18, Lemma 5.1], for every $t \in \mathbb{T} \setminus \{0\}$, $\mathbb{P}$ is ergodic with respect to θ^t.

Let (X, d) be a separable metric space such that X is a Borel subset of the d-completion of X.[5] For any $x \in X$ and $\delta > 0$, we write $B_\delta(x) := \{y \in X : d(x, y) < \delta\}$. For any $A \subset X$, we write $\Delta_A := \{(x, x) : x \in A\} \subset X \times X$.

Let $\varphi = (\varphi(t, \omega))_{t\in\mathbb{T}^+,\,\omega\in\Omega}$ be a $(\mathbb{T}^+ \times \Omega)$-indexed family of continuous functions $\varphi(t, \omega) : X \to X$ such that

(a) the map $\omega \mapsto \varphi(t, \omega)x$ is $(\mathcal{F}_t, \mathcal{B}(X))$-measurable for each $t \in \mathbb{T}^+$ and $x \in X$;

(b) for every $\omega \in \Omega$, $\varphi(0, \omega)$ is the identity function on X;

(c) $\varphi(s + t, \omega) = \varphi(t, \theta^s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{T}^+$ and $\omega \in \Omega$;

(d) for any decreasing sequence (t_n) in $\mathbb{T}^+$ converging to a value t, and any sequence (x_n) in X converging to a point x, $\varphi(t_n, \omega)x_n \to \varphi(t, \omega)x$ as $n \to \infty$ for all $\omega \in \Omega$.

(Property (d) constitutes "right-continuity" of φ.)

We refer to φ as a *random dynamical system on the state space X over the noise space* $(\Omega, \mathcal{F}, (\mathcal{F}_s^{s+t})_{s\in\mathbb{T},\,t\in\mathbb{T}^+}, \mathbb{P}, (\theta^t)_{t\in\mathbb{T}})$.

Now it is easy to show that for any $x \in X$, the stochastic process $(\varphi(t, \cdot)x)_{t\in\mathbb{T}^+}$ is a homogeneous Markov process (with respect to the filtration $(\mathcal{F}_0^t)_{t\in\mathbb{T}^+}$), with the associated family of transition probabilities $(\varphi_x^t)_{x\in X,\,t\in\mathbb{T}^+}$ being given by

$$\begin{aligned}
\varphi_x^t(A) &:= \mathbb{P}(\omega : \varphi(t, \omega)x \in A) \\
&= \mathbb{P}(\omega : \varphi(t, \theta^s\omega)x \in A) \quad \text{(for any } s \in \mathbb{T})
\end{aligned}$$

for all $A \in \mathcal{B}(X)$. Note that a probability measure ρ on X is a stationary probability measure of the Markov transition probabilities $(\varphi_x^t)_{x\in X,\,t\in\mathbb{T}^+}$ if and only if for all $A \in \mathcal{B}(X)$ and $t \in \mathbb{T}^+$,

$$\rho(A) = \int_\Omega \rho(\varphi(t, \omega)^{-1}(A))\, \mathbb{P}(d\omega).$$

For any $t \in \mathbb{T}^+$ we define the map $\Theta^t : \Omega \times X \to \Omega \times X$ by

$$\Theta^t(\omega, x) = (\theta^t\omega, \varphi(t, \omega)x).$$

[5] This guarantees that X is measurably isomorphic to either an at-most-countable discrete space, or an interval with its Borel σ-algebra ([21, Theorem 3.3.13]).

Note that $(\Theta^t)_{t\in\mathbb{T}^+}$ forms a semigroup of measurable transformations of the measurable space $(\Omega\times X, \mathcal{F}\otimes\mathcal{B}(X))$, and also of the "restricted" measurable space $(\Omega\times X, \mathcal{F}_{-r}^{\infty}\otimes\mathcal{B}(X))$ for any $r\in\mathbb{T}^+$. For any Borel probability measure ρ on X, the following hold:

- ρ is a stationary measure of the Markov transition probabilities $(\varphi_x^t)_{x\in X,\, t\in\mathbb{T}^+}$ if and only if $(\Theta^t)_{t\in\mathbb{T}^+}$ is a measure-preserving semigroup of the probability space $(\Omega\times X, \mathcal{F}_0^{\infty}\otimes\mathcal{B}(X), \mathbb{P}|_{\mathcal{F}_0^\infty}\otimes\rho)$;
- ρ is an ergodic measure of the Markov transition probabilities $(\varphi_x^t)_{x\in X,\, t\in\mathbb{T}^+}$ if and only if $(\Theta^t)_{t\in\mathbb{T}^+}$ is an ergodic measure-preserving semigroup of the probability space $(\Omega\times X, \mathcal{F}_0^{\infty}\otimes\mathcal{B}(X), \mathbb{P}|_{\mathcal{F}_0^\infty}\otimes\rho)$.

(For a proof, see e.g. [17, Theorem 143] or [12, Lemma I.2.3 and Theorem I.2.1].)

2.2. **Stability of trajectories and our main result.** We now introduce the notion of asymptotic stability; we then give (a generalised version of) an important result in [14], and from there, state our main result.

Given a sample point $\omega\in\Omega$ and a set $A\subset X$, we say that A *contracts under* ω if $\mathrm{diam}(\varphi(t,\omega)A)\to 0$ as $t\to\infty$. Given a sample point $\omega\in\Omega$ and a point $x\in X$, we say that *x is asymptotically stable under ω* if there exists a neighbourhood U of x such that U contracts under ω. We say that a set $A\subset X$ *admits stable trajectories* if

$$\mathbb{P}(\omega \,:\, \exists\,\text{open } U \text{ with } U\cap A\neq\emptyset \text{ s.t. } U \text{ contracts under } \omega) \;>\; 0,$$

which is the same as saying that

$$\mathbb{P}(\omega \,:\, \exists\, x\in A \text{ s.t. } x \text{ is asymptotically stable under } \omega) \;>\; 0.$$

Now let

$$O \;:=\; \{(\omega,x)\in\Omega\times X : x \text{ is asymptotically stable under } \omega\}.$$

As in [19, Lemma 3.2.3], O is an $(\mathcal{F}_0^\infty\otimes\mathcal{B}(X))$-measurable set, and is backward-invariant under the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$.

Lemma 2.1. *Let ρ be an ergodic probability measure of the Markov transition probabilities (φ_x^t). The following statements are equivalent:*

(i) O is a $(\mathbb{P}\otimes\rho)$-full measure set;
(ii) O is a $(\mathbb{P}\otimes\rho)$-positive measure set;
(iii) $\mathrm{supp}\,\rho$ *admits stable trajectories.*

Proof. The equivalence of (i) and (ii) follows from the backward-invariance of O and the fact that $\mathbb{P}|_{\mathcal{F}_0^\infty}\otimes\rho$ is (Θ^t)-ergodic. It is clear that (ii)⇒(iii). Now suppose that (iii) holds; so we have a $\mathbb{P}$-positive measure set of sample points ω with the property that there exists a ρ-positive measure open set U such that U contracts under ω. Fubini's theorem then yields that $\mathbb{P}\otimes\rho(O)>0$, i.e. (ii) holds. □

Definition 2.2. Let ρ be an ergodic probability measure of (φ_x^t). We say that *φ is stable with respect to ρ* if the equivalent statements in Lemma 2.1 hold.

Now given a sample point $\omega\in\Omega$ and an open set $U\subset X$, we will say that *U is σ-contracting under ω* if there exists an increasing sequence $U_1\subset U_2\subset U_3\subset\ldots$ of open subsets of X such that $U=\bigcup_{i=1}^\infty U_i$ and U_i contracts under ω for each $i\in\mathbb{N}$.

Definition 2.3. Let ρ be an ergodic probability measure of (φ_x^t). We say that *φ is ρ-almost everywhere stably synchronising* if for $\mathbb{P}$-almost every $\omega\in\Omega$ there exists a ρ-full measure open set that is σ-contracting under ω.

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Remark 2.4. It is not hard to show that φ is ρ-almost everywhere stably synchronising if and only if the following statements both hold:

(i) φ is stable with respect to ρ;
(ii) there is a ρ-full set $A \subset X$ such that for all $x, y \in A$,

$$\mathbb{P}(\omega \,:\, d(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0 \text{ as } t \to \infty) \;=\; 1.$$

The following is a generalised statement of [14, Proposition 3]:

Proposition 2.5. *Let ρ be an ergodic probability measure of (φ_x^t), and suppose that φ is stable with respect to ρ. Then there exists $n_\rho \in \mathbb{N}$ such that the following holds: for $\mathbb{P}$-almost every $\omega \in \Omega$, there exist open sets $U_1(\omega), \ldots, U_{n_\rho}(\omega)$ with $\rho(U_i(\omega)) = \frac{1}{n_\rho}$ for each $1 \leq i \leq n_\rho$ such that*

- *$U_i(\omega)$ is σ-contracting under ω for each $1 \leq i \leq n_\rho$, and*
- *for every $1 \leq i < j \leq n_\rho$, for every $x \in U_i(\omega)$ and $y \in U_j(\omega)$, $d(\varphi(t,\omega)x, \varphi(t,\omega)y)$ does not tend to 0 as $t \to \infty$.*

(The proof will be outlined in Section 3.)

So the situation that φ is ρ-almost everywhere stably synchronising is precisely the situation that $n_\rho = 1$. We go on to present our new sharp criteria for this situation.

Definition 2.6. Given points $x, y, p \in X$, we will say that (x, y) *is contractible towards* p if for every $\varepsilon > 0$,

$$\mathbb{P}(\,\omega \,:\, \exists\, t \in \mathbb{T}^+ \text{ s.t. } (\varphi(t,\omega)x, \varphi(t,\omega)y) \in B_\varepsilon(p) \times B_\varepsilon(p)\,) \;>\; 0.$$

Since the map $t \mapsto \varphi(t,\omega)u$ is right-continuous for all $u \in X$, this is equivalent to saying that there exists $t \in \mathbb{T}^+ \cap \mathbb{Q}$ such that

$$\mathbb{P}(\,\omega \,:\, (\varphi(t,\omega)x, \varphi(t,\omega)y) \in B_\varepsilon(p) \times B_\varepsilon(p)\,) \;>\; 0.$$

Definition 2.7. Given points $x, y \in X$ and a set $A \subset X$, we will say that (x, y) *is contractible towards* A if for every neighbourhood V of Δ_A in $X \times X$,

$$\mathbb{P}(\,\omega \,:\, \exists\, t \in \mathbb{T}^+ \text{ s.t. } (\varphi(t,\omega)x, \varphi(t,\omega)y) \in V\,) \;>\; 0.$$

Lemma 2.8. *For any $x, y \in X$ and $A \subset X$, (x, y) is contractible towards A if and only if there exists $p \in A$ such that (x, y) is contractible towards p.*

Proof. It is clear that if there exists $p \in A$ such that (x, y) is contractible towards p, then (x, y) is contractible towards A. Now suppose there does not exist $p \in A$ such that (x, y) is contractible towards p. Let

$$\begin{aligned}\mathcal{U} \;&:=\; \{\,\text{open } V \subset X \times X \,:\, \mathbb{P}(\,\omega \,:\, \exists\, t \in \mathbb{T}^+ \text{ s.t. } (\varphi(t,\omega)x, \varphi(t,\omega)y) \in V\,) \;=\; 0\,\} \\ &=\; \{\,\text{open } V \subset X \times X \,:\, \text{for all } t \in \mathbb{T}^+ \cap \mathbb{Q},\ \mathbb{P}(\,\omega \,:\, (\varphi(t,\omega)x, \varphi(t,\omega)y) \in V\,) \;=\; 0\,\}\end{aligned}$$

and let $W := \bigcup_{V \in \mathcal{U}} V$. For every $p \in A$, since (x, y) is not contractible towards p, there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \times B_\varepsilon(p) \subset W$. Hence $\Delta_A \subset W$. Now since $X \times X$ is second-countable, there exists a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $W = \bigcup_{V \in \mathcal{V}} V$. It therefore follows in particular that for every $t \in \mathbb{T}^+ \cap \mathbb{Q}$,

$$\mathbb{P}(\,\omega \,:\, (\varphi(t,\omega)x, \varphi(t,\omega)y) \in W\,) \;=\; 0.$$

Hence

$$\mathbb{P}(\,\omega \,:\, \exists\, t \in \mathbb{T}^+ \text{ s.t. } (\varphi(t,\omega)x, \varphi(t,\omega)y) \in W\,) \;=\; 0.$$

So (x, y) is not contractible towards A. □

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Definition 2.9. Let ρ be an ergodic probability measure of (φ_x^t). We will say that a point $x \in \operatorname{supp}\rho$ is *ρ-transitive* if for every open $U \subset X$ with $\rho(U) > 0$,

$$\mathbb{P}(\,\omega \,:\, \exists\, t \in \mathbb{T}^+ \text{ s.t. } \varphi(t,\omega)x \in U\,) \;>\; 0.$$

This is equivalent to saying that for some $t \in \mathbb{T}^+ \cap \mathbb{Q}$, $\varphi_x^t(U) > 0$.

We will use the following notations:

- For any $p \in X$, $\mathfrak{C}_p \subset X \times X$ denotes the set of pairs that are contractible towards p.
- For any $A \subset X$, $\mathfrak{C}_A \subset X \times X$ denotes the set of pairs that are contractible towards A. In other words (by Lemma 2.8), $\mathfrak{C}_A = \bigcup_{p\in A} \mathfrak{C}_p$.
- For any ergodic probability measure ρ of (φ_x^t), A_ρ denotes the set of points in $\operatorname{supp}\rho$ that are ρ-transitive.

By the ergodic theorem for Markov processes,[6] ρ-almost every $x \in \operatorname{supp}\rho$ has the property that for $\mathbb{P}$-almost all $\omega \in \Omega$, for every $T \in \mathbb{T}^+$, $\{\varphi(t,\omega)x : t \geq T\}$ is dense in $\operatorname{supp}\rho$. Hence in particular, $\rho(A_\rho) = 1$.

Definition 2.10. Let ρ be a probability measure on X. A *ρ-full-length rectangle* is a set $A \subset X \times X$ taking the form $A = A_1 \times A_2$ where $A_1, A_2 \in \mathcal{B}(X)$ with $\rho(A_1) > 0$ and $\rho(A_2) = 1$.

Our main result is the following:

Theorem 2.11. *Let ρ be an ergodic probability measure of (φ_x^t), and suppose that φ is stable with respect to ρ. The following statements are equivalent:*

(i) there is a non-ρ-null set $R \subset X$ such that for each $p \in R$, the set $\mathfrak{C}_p$ contains a ρ-full-length rectangle;
(ii) the set $\mathfrak{C}_{A_\rho}$ contains a ρ-full-length rectangle;
(iii) φ is ρ-almost everywhere stably synchronising;
(iv) there is a ρ-full set $A \subset \operatorname{supp}\rho$ such that given any $x, y \in A$, $\mathbb{P}$-almost every $\omega \in \Omega$ has the property that for any open $U \subset X$ with $\rho(U) > 0$ there exists $t \in \mathbb{T}^+$ such that $\varphi(t,\omega)x, \varphi(t,\omega)y \in U$.

Let us now consider the example of a "double-well potential perturbed by Gaussian white noise". Fix an integer $d \geq 2$. Let $\mathbb{T} = \mathbb{R}$. Let $\Omega := \{\omega \in C(\mathbb{R},\mathbb{R}^d) : \omega(0) = \mathbf{0}\}$, let $\mathcal{F}$ be the smallest σ-algebra on Ω with respect to which the projections $\omega \mapsto \omega(t)$ are measurable for all $t \in \mathbb{R}$, let $\mathbb{P}$ be the Wiener measure on $(\Omega,\mathcal{F})$, and for each $\tau \in \mathbb{R}$ let $\theta^\tau \colon \Omega \to \Omega$ be given by $(\theta^\tau\omega)(t) = \omega(t+\tau) - \omega(\tau)$. Let $X = \mathbb{R}^d$ (equipped with the Euclidean metric). As in [7], let φ be such that for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$, the function $u(t) = \varphi(t,\omega)x$ is the solution of the integral equation

$$\begin{aligned} u(t) \;&=\; x + \int_0^t (1 - |u(s)|^2)u(s)\,ds + \omega(t) \\ &=\; x + \int_0^t b(u(s))\,ds + \omega(t) \end{aligned}$$

where $b(y) := (1 - |y|^2)y$ for all $y \in \mathbb{R}^d$. In other words, φ is the "RDS generated by the stochastic differential equation"

$$du_t \;=\; (1 - |u_t|^2)u_t\,dt \;+\; dW_t.$$

[6] See e.g. [17, Corollary 57], with Y being the set of right-continuous paths in X.

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It is not hard to show (e.g. by computing explicitly the Jacobian of b) that, as in [7], b satisfies the *one-sided Lipschitz condition*—that is to say, there exists $L \in \mathbb{R}$ such that for all $y_1, y_2 \in \mathbb{R}^d$,

$$(b(y_2) - b(y_1)) \cdot (y_2 - y_1) \;\leq\; L|y_2 - y_1|^2.$$

Now for any $\omega \in \Omega$, $x_1, x_2 \in \mathbb{R}^d$ and $t_0 \in \mathbb{R}$, if we let $u_1(t) := \varphi(t, \theta^{t_0}\omega)x_1$ and $u_2(t) := \varphi(t, \theta^{t_0}\omega)x_2$ for all $t \geq 0$, we find that

$$u_2(t) - u_1(t) \;=\; (u_2(0) - u_1(0)) + \int_0^t b(u_2(s)) - b(u_1(s))\, ds$$

and therefore

$$\frac{d}{d\tau}|u_2(\tau) - u_1(\tau)|^2\bigg|_{\tau=t} \;=\; 2(b(u_2(t)) - b(u_1(t))) \cdot (u_2(t) - u_1(t)) \;\leq\; 2L|u_2(t) - u_1(t)|^2.$$

Grönwall's inequality then gives that

$$|u_2(t) - u_1(t)| \;\leq\; |u_2(0) - u_1(0)|e^{Lt}.$$

In other words, for all $\omega \in \Omega$, $x_1, x_2 \in \mathbb{R}^d$, and $t_0, t_1 \in \mathbb{R}$ with $t_1 \geq t_0$, we have

$$|\varphi(t_1, \omega)x_2 \,-\, \varphi(t_1, \omega)x_1| \;\leq\; e^{L(t_1 - t_0)}|\varphi(t_0, \omega)x_2 \,-\, \varphi(t_0, \omega)x_1|.$$

As a consequence, we have that for any $A \subset \mathbb{R}^d$ and any $\omega \in \Omega$,

$$\operatorname{diam}(\varphi(n, \omega)A) \to 0 \text{ as } n \to \infty \text{ in } \mathbb{N} \quad \Longrightarrow \quad A \text{ contracts under } \omega. \tag{2.1}$$

Now as in [7], there exists a unique (φ_x^t)-ergodic probability measure ρ on $\mathbb{R}^d$, and ρ has full support. By [7, Example 3.8], the maximal Lyapunov exponent associated to ρ is negative. By results in [7, Section 3.1], it follows that for $(\mathbb{P} \otimes \rho)$-almost all $(\omega, x) \in \Omega \times \mathbb{R}^d$, there is a neighbourhood U of x such that $\operatorname{diam}(\varphi(n, \omega)U) \to 0$ as $n \to \infty$ in $\mathbb{N}$; so (2.1) then gives that φ is stable with respect to ρ. Now (as with any additive-noise SDE) one can show that every point in $\mathbb{R}^d$ is ρ-transitive: Fix any $x \in \mathbb{R}^d$ and any non-empty open $U \subset \mathbb{R}^d$; take any $y \in U$ and, selecting a sufficiently large value $\eta_0 > 0$, take a sample point $\omega_0 \in \Omega$ with

$$\omega_0(t) \;=\; \eta_0 t(y - x) \quad \forall\, t \in [0, \tfrac{1}{\eta_0}].$$

Then we will have that $\varphi(\frac{1}{\eta_0}, \omega_0) \in U$. Since the Wiener measure $\mathbb{P}$ has full support in the topology of uniform convergence on compact sets ([9, Proposition 477F]), it follows that $\varphi_x^{1/\eta_0}(U) > 0$. Since U was arbitrary, x is ρ-transitive. Now it is not hard to see that every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ is contractible towards the point $(1, \mathbf{0}) \in \mathbb{R}^d$: Fixing any $\varepsilon > 0$, we can select sufficiently large values $\eta_1, \eta_2 > 0$ that if we take a sample point ω_1 with

$$\omega_1(t) \;=\; \begin{cases} (\eta_1\eta_2 t, \mathbf{0}) & t \in [0, \frac{1}{\eta_1}] \\ (\eta_2, \mathbf{0}) & t \in [\frac{1}{\eta_1}, \infty), \end{cases}$$

we will have that $\varphi(t, \omega_1)x,\, \varphi(t, \omega_1)y \in B_\varepsilon((1, \mathbf{0}))$ for all sufficiently large t; so once again, since $\mathbb{P}$ has full support, it follows that (x, y) is contractible towards $(1, \mathbf{0})$. So then, φ satisfies hypothesis (ii) of Theorem 2.11 (since $A_\rho = \mathbb{R}^d$ and $\mathfrak{C}_{\mathbb{R}^d} \supset \mathfrak{C}_{(1,\mathbf{0})} = \mathbb{R}^d \times \mathbb{R}^d$), and therefore φ is ρ-almost stably synchronising. Now by Remark 2.4, there exists a ρ-full set $A \subset X$ such that for all $x \in A$,

$$\mathbb{P}(\omega \,:\, x \text{ is asymptotically stable under } \omega) \;=\; 1$$

and for all $x, y \in A$,

$$\mathbb{P}(\omega : d(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0 \text{ as } t \to \infty) = 1.$$

Now for every $x \in X$ and $t > 0$, φ_x^t is equivalent to the Lebesgue measure, and also the stationary measure ρ is equivalent to the Lebesgue measure. Hence, for every $x \in X$ we have that $\varphi_x^1(A) = 1$, and for all $x, y \in X$ we have that

$$\mathbb{P}(\omega : \varphi(1,\omega)x, \varphi(1,\omega)y \in A) = 1.$$

Consequently, due to the memorylessness of the noise, we can conclude that for *all* $x \in X$,

$$\mathbb{P}(\omega : x \text{ is asymptotically stable under } \omega) = 1,$$

and for *all* $x, y \in X$,

$$\mathbb{P}(\omega : d(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0 \text{ as } t \to \infty) = 1.$$

3. Invariant measures and the proof of Theorem 2.11

We start by introducing some basic theory of invariant measures of random dynamical systems. We define the projections $\pi_\Omega : \Omega \times X \to \Omega$ and $\pi_X : \Omega \times X \to X$ by $\pi_\Omega(\omega, x) = \omega$ and $\pi_X(\omega, x) = x$.

A *random probability measure on* X is an Ω-indexed family $(\mu_\omega)_{\omega\in\Omega}$ of probability measures on X such that the map $\omega \mapsto \mu_\omega(A)$ is measurable for all $A \in \mathcal{B}(X)$. We will say that two random probability measures $(\mu_\omega^1)_{\omega\in\Omega}$ and $(\mu_\omega^2)_{\omega\in\Omega}$ are *equivalent* if for $\mathbb{P}$-almost all $\omega \in \Omega$, $\mu_\omega^1 = \mu_\omega^2$. For any random probability measure (μ_ω), we may define a probability measure μ on the product space $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ by

$$\mu(A) = \int_\Omega \mu_\omega(A_\omega)\, \mathbb{P}(d\omega) \quad \forall\, A \in \mathcal{B}(X)$$

where $A_\omega := \{x \in X : (\omega, x) \in A\}$. We refer to μ as the *integrated form of* (μ_ω). Note that two equivalent random probability measures share the same integrated form.

We will say that a random probability measure (μ_ω) is *atomless* if for $\mathbb{P}$-almost every $\omega \in \Omega$, μ_ω is an atomless probability measure (i.e. $\mu_\omega(\{x\}) = 0$ for all $x \in X$). Given an integer $n \in \mathbb{N}$, we will say that a random probability measure (μ_ω) is *n-uniform* if for $\mathbb{P}$-almost every $\omega \in \Omega$ there exist distinct points $x_1, \ldots, x_n \in X$ such that $\mu_\omega = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$.

Remark 3.1. Given a random probability measure (μ_ω), let $\bar{\mu}$ be the probability measure on X given by

$$\bar{\mu}(A) = \int_\Omega \mu_\omega(A)\, \mathbb{P}(d\omega)$$

for all $A \in \mathcal{B}(X)$, and let $\bar{\mu}^{(2)}$ be the probability measure on $X \times X$ given by

$$\bar{\mu}^{(2)}(A) = \int_\Omega \mu_\omega \otimes \mu_\omega(A)\, \mathbb{P}(d\omega)$$

for all $A \in \mathcal{B}(X \times X)$. (Note that $\bar{\mu}$ is precisely $\pi_{X*}\mu$, where μ is the integrated form of (μ_ω).) By Fubini's theorem, if (μ_ω) is atomless then $\bar{\mu}^{(2)}(\Delta_X) = 0$, and if (μ_ω) is n-uniform then for all $A \in \mathcal{B}(X)$, $\bar{\mu}^{(2)}(\Delta_A) = \frac{1}{n}\bar{\mu}(A)$.

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Now we will say that a probability measure μ on the product space $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ is $\mathbb{P}$-*compatible* if $\pi_{\Omega *}\mu = \mathbb{P}$. It is clear that the integrated form of a random probability measure is itself a $\mathbb{P}$-compatible probability measure. The *disintegration theorem* ([5, Proposition 3.6]) states that for any $\mathbb{P}$-compatible probability measure μ there exists a random probability measure (μ_ω) whose integrated form coincides with μ, and this random probability measure is unique up to equivalence; we refer to (μ_ω) as *a (version of the) disintegration* of μ. We say that a $\mathbb{P}$-compatible probability measure μ is *past-measurable* if μ admits a disintegration (μ_ω) such that the map $\omega \mapsto \mu_\omega(A)$ is $\mathcal{F}_{-\infty}^0$-measurable for all $A \in \mathcal{B}(X)$. It is easy to show (using the fact that $\mathcal{F}_{-\infty}^0$ and $\mathcal{F}_0^\infty$ are independent σ-algebras) that for any past-measurable $\mathbb{P}$-compatible probability measure μ, the restriction of μ to $\mathcal{F}_0^\infty \otimes \mathcal{B}(X)$ coincides with $\mathbb{P}|_{\mathcal{F}_0^\infty} \otimes \pi_{X*}\mu$.

We will say that a probability measure μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ is *an invariant measure of* φ if μ is both $\mathbb{P}$-compatible and invariant under the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$. It is not hard to show that a $\mathbb{P}$-compatible probability measure μ with disintegration (μ_ω) is invariant under φ if and only if

$$\mathbb{P}(\omega \in \Omega : \mu_{\theta^t\omega} = \varphi(t,\omega)_*\mu_\omega) = 1 \quad \forall\, t \in \mathbb{T}^+. \tag{3.1}$$

We will say that a probability measure μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ is *an ergodic measure of* φ if μ is both $\mathbb{P}$-compatible and ergodic with respect to the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$.

The following is essentially part (a) of the proof of [14, Proposition 2]:

Proposition 3.2. *Let μ be an ergodic measure of φ, and let (μ_ω) be a disintegration of μ. Then either (μ_ω) is atomless or there exists $n \in \mathbb{N}$ such that (μ_ω) is n-uniform.*

Proof. Define the function $h\colon \Omega \times X \to [0,1]$ by $h(\omega,x) = \mu_\omega(\{x\})$. Note that h is measurable, since it can be expressed as

$$h(\omega,x) \;=\; \int_X \mathbb{1}_{\Delta_X}(x,y)\,\mu_\omega(dy).$$

Now for each $t \in \mathbb{T}^+$, let $\Omega_t \subset \Omega$ be a $\mathbb{P}$-full set such that for each $\omega \in \Omega_t$, $\mu_{\theta^t\omega} = \varphi(t,\omega)_*\mu_\omega$. Then for all $(\omega,x) \in \Omega_t \times X$, we have

$$\begin{aligned} h(\Theta^t(\omega,x)) &= \mu_{\theta^t(\omega)}(\,\{\varphi(t,\omega)x\}\,) \\ &= \mu_\omega\left(\,\varphi(t,\omega)^{-1}(\,\{\varphi(t,\omega)x\}\,)\,\right) \\ &\geq \mu_\omega(\{x\}) \\ &= h(\omega,x). \end{aligned}$$

Since μ is $\mathbb{P}$-compatible, $\mu(\Omega_t \times X) = 1$ and so $h \circ \Theta^t \overset{\mu\text{-a.s.}}{\geq} h$ for each $t \in \mathbb{T}^+$. Hence, since μ is (Θ^t)-ergodic, there exists $c \in [0,1]$ such that $h^{-1}(\{c\})$ is a μ-full set. So for $\mathbb{P}$-almost every $\omega \in \Omega$, μ_ω has the property that $\mu_\omega(\{x\}) = c$ for μ_ω-almost all $x \in X$. It is then clear that either $c = 0$ and (μ_ω) is atomless, or $c = \frac{1}{n}$ for some $n \in \mathbb{N}$ and (μ_ω) is n-uniform. □

Now let $\mathcal{S}$ be the set of probability measures on X that are stationary with respect to the Markov transition probabilities (φ_x^t). Let $\mathcal{I}$ be the set of past-measurable invariant measures of φ. The following is [13, Theorem 4.2.9]:[7]

[7] In [13], it is assumed that X is Polish, allowing in particular for the result of [4] to be applied in the construction of the random measure μ_ω. Nonetheless, in the more general case that X is separable and is Borel in the d-completion of X, one can regard X (topologically) as a measurable

Proposition 3.3. *$\mathcal{I}$ is mapped bijectively into $\mathcal{S}$ by the mapping $\mathfrak{r}\colon \mu \mapsto \pi_{X*}\mu$. For any $\rho \in \mathcal{S}$, letting (μ_ω) be a disintegration of the past-measurable invariant measure $\mathfrak{r}^{-1}(\rho)$, we have that for any unbounded increasing sequence (t_n) in $\mathbb{T}^+$ there exists a $\mathbb{P}$-full set $\tilde{\Omega} \subset \Omega$ such that for all $\omega \in \tilde{\Omega}$, $\varphi(t_n, \theta^{-t_n}\omega)_*\rho$ converges weakly to μ_ω as $n \to \infty$.*

In addition, we have the following:

Proposition 3.4. *For any $\rho \in \mathcal{S}$, letting μ_ρ denote the unique past-measurable invariant measure of φ satisfying $\pi_{X*}\mu_\rho = \rho$, μ_ρ is also the only (Θ^t)-invariant probability measure on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$ whose restriction to $\mathcal{F}_0^\infty \otimes \mathcal{B}(X)$ coincides with $\mathbb{P}|_{\mathcal{F}_0^\infty} \otimes \rho$.*

Proof. Fix $\rho \in \mathcal{S}$. Let μ' be any (Θ^t)-invariant probability measure with the property that $\mu'|_{\mathcal{F}_0^\infty \otimes \mathcal{B}(X)} = \mathbb{P}|_{\mathcal{F}_0^\infty} \otimes \rho$. Note that for each $t \in \mathbb{T}^+$, Θ^t is $(\mathcal{F}_0^\infty \otimes \mathcal{B}(X), \mathcal{F}_{-t}^\infty \otimes \mathcal{B}(X))$-measurable; so then, for any $t \in \mathbb{T}^+$, for all $A \in \mathcal{F}_{-t}^\infty \otimes \mathcal{B}(X)$,

$$\mu'(A) \;=\; \mu'(\Theta^{-t}(A)) \;=\; \mathbb{P} \otimes \rho(\Theta^{-t}(A)) \;=\; \mu_\rho(\Theta^{-t}(A)) \;=\; \mu_\rho(A).$$

Since μ' and μ_ρ agree on $\mathcal{F}_{-t}^\infty \otimes \mathcal{B}(X)$ for all $t \in \mathbb{T}^+$ and (by assumption) $\mathcal{F}$ is the σ-algebra generated by $\bigcup_{t \in \mathbb{T}^+} \mathcal{F}_{-t}^\infty$, it follows that μ' and μ_ρ agree on the whole of $\mathcal{F} \otimes \mathcal{B}(X)$. □

As an immediate consequence of Propositions 3.3 and 3.4, we have:

Corollary 3.5. *For any $\mu \in \mathcal{I}$, μ is an ergodic measure of φ if and only if $\pi_{X*}\mu$ is ergodic with respect to (φ_x^t).*

Remark 3.6. The above one-to-one correspondence between past-measurable invariant measures and stationary probability measures is a particular case of a more general one-to-one correspondence between invariant measures and "forward-time invariant measures", as described in [2, Theorem 1.7.2].

Now we define the *two-point motion* $\varphi \times \varphi = (\varphi{\times}\varphi(t,\omega))_{t \in \mathbb{T}^+,\, \omega \in \Omega}$ to be the $(\mathbb{T}^+ \times \Omega)$-indexed family of functions $\varphi{\times}\varphi(t,\omega) : X \times X \to X \times X$ given by

$$\varphi{\times}\varphi(t,\omega)(x,y) \;=\; (\varphi(t,\omega)x, \varphi(t,\omega)y).$$

Note that $\varphi \times \varphi$ is itself a random dynamical system on $X \times X$. We may define the associated family of Markov transition probabilities $(\varphi_{(x,y)}^t)_{x,y \in X,\, t \in \mathbb{T}^+}$ by

$$\varphi_{(x,y)}^t(A) \;=\; \mathbb{P}(\omega : (\varphi(t,\omega)x, \varphi(t,\omega)y) \in A) \quad \forall\, A \in \mathcal{B}(X \times X).$$

Remark 3.7. For any invariant measure μ of φ, letting (μ_ω) be a disintegration of μ, the integrated form of $(\mu_\omega \otimes \mu_\omega)_{\omega \in \Omega}$ is an invariant measure of $\varphi \times \varphi$; so if the invariant measure μ is past-measurable then (as in [3, Proposition 2.6(ii)]) the measure $\bar{\mu}^{(2)}$ as defined in Remark 3.1 is a stationary probability measure of $(\varphi_{(x,y)}^t)$.

Standing Assumption. *From now on, fix a (φ_x^t)-ergodic probability measure ρ, and (on the basis of Proposition 3.3 and Corollary 3.5) let μ be the unique past-measurable ergodic measure of φ satisfying $\pi_{X*}\mu = \rho$, and let (μ_ω) be a disintegration*

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subset of the compact space $[0,1]^{\mathbb{N}}$; one can then construct the random measure $\tilde{\mu}_\omega$ on $[0,1]^{\mathbb{N}}$ as the almost sure limit of the sequence of random measures $\mu_\omega^{(n)}(\cdot) := \rho(\varphi(n,\omega)^{-1}(\,\cdot \cap X))$, and then (since $\mathbb{E}[\tilde{\mu}_\omega] = \rho(\,\cdot \cap X)$) one can take μ_ω to be the restriction of $\tilde{\mu}_\omega$ to $\mathcal{B}(X)$.

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of μ. Let $\bar{\mu}^{(2)}$ be the associated $(\varphi^t_{(x,y)})$-stationary probability measure on $X \times X$ as described in Remark 3.7. For each $\omega \in \Omega$, we define the equivalence relation $\sim_\omega$ on X by

$$x \sim_\omega y \quad \Longleftrightarrow \quad d(\,\varphi(t,\omega)x\,,\,\varphi(t,\omega)y\,) \;\to\; 0 \;\; \text{as } t \to \infty.$$

Let us now outline the proof of Proposition 2.5: It is not hard to show that if φ is stable with respect to ρ, then $\bar{\mu}^{(2)}(\Delta_X) > 0$ and therefore (μ_ω) is not atomless;[8] so by Proposition 3.2 there exists $n_\rho \in \mathbb{N}$ such that (μ_ω) is n_ρ-uniform. So $A(\omega) := \operatorname{supp}\mu_\omega$ is almost surely an n_ρ-element set. Due to (3.1) and the (θ^t)-invariance of $\mathbb{P}$, we have that for $\mathbb{P}$-almost all ω the elements of $A(\omega)$ belong to distinct equivalence classes of $\sim_\omega$. Since φ is stable with respect to ρ, we have that for $\mathbb{P}$-almost all ω, for each $x \in A(\omega)$, the $\sim_\omega$-equivalence class of x contains a neighbourhood of x. Consequently, as in [14, Proposition 3], one can use the construction of $\mathfrak{r}^{-1}$ in Proposition 3.3 together with the (θ^t)-invariance of $\mathbb{P}$ to deduce that $\mathbb{P}$-almost all ω, for each $x \in A(\omega)$, the $\sim_\omega$-equivalence class of x contains an open set of measure $\frac{1}{n_\rho}$ under ρ. (By the second-countability of X and the fact that φ is stable with respect to ρ, this open set is σ-contracting under ω.)

We now prove our main result:

Proof of Theorem 2.11. Suppose (i) holds; then since A_ρ is a ρ-full set, $A_\rho \cap R \neq \emptyset$, and so there exists $p \in A_\rho$ such that $\mathfrak{C}_p$ contains a ρ-full-length rectangle, implying (ii).

Now suppose that (ii) holds. For each $t \in \mathbb{T}^+$, define the map $\Theta^t_{[2]} \colon \Omega \times X \times X \to \Omega \times X \times X$ by

$$\Theta^t_{[2]}(\omega,x,y) \;:=\; (\theta^t\omega, \varphi(t,\omega)x, \varphi(t,\omega)y).$$

Note that the probability measure $\mathbb{P}|_{\mathcal{F}_0^\infty} \otimes \bar{\mu}^{(2)}$ on $(\Omega \times X \times X, \mathcal{F}_0^\infty \otimes \mathcal{B}(X \times X))$ is invariant under the semigroup $(\Theta^t_{[2]})_{t \in \mathbb{T}^+}$. So by the Poincaré recurrence theorem,

$$\mathbb{P} \otimes \bar{\mu}^{(2)}((\omega,x,y) : x \neq y,\, x \sim_\omega y) \;=\; 0.$$

Hence, by Fubini's theorem, the set

$$Y \;:=\; \{\, (x,y) \in (X \times X) \setminus \Delta_X \,:\, \mathbb{P}(\omega : x \sim_\omega y) > 0 \,\}$$

is an $\bar{\mu}^{(2)}$-null set. Now let $A_1, A_2 \in \mathcal{B}(X)$ be such that $\rho(A_1) > 0$, $\rho(A_2) = 1$ and $A_1 \times A_2 \subset \mathfrak{C}_{A_\rho}$. We will show that for any $(x,y) \in A_1 \times A_2$, $\mathbb{P}(\omega : x \sim_\omega y) > 0$. Fix any $(x,y) \in A_1 \times A_2$, and let $p \in A_\rho$ be such that (x,y) is contractible towards p. Taking any $z \in \operatorname{supp}\rho$ for which $\mathbb{P}(\omega : z \text{ is asymptotically stable under } \omega) > 0$, there must exist an open neighbourhood V of z such that $\mathbb{P}(E_V) > 0$, where we define $E_V := \{\omega : V \text{ contracts under } \omega\}$; so take such a neighbourhood V of z, and take an open neighbourhood U of z such that $\bar{U} \subset V$. Since p is ρ-transitive, let $t_1 \in \mathbb{T}^+$ be such that $\varphi_p^{t_1}(U) > 0$. Since $\varphi(t_1,\omega)$ is continuous for all ω, let $r > 0$ be such that

$$k_1 := \mathbb{P}(\omega : \varphi(t_1,\omega)\overline{B_r(p)} \subset \bar{U}) \;>\; 0$$

and let $t_0 \in \mathbb{T}^+$ be such that

$$k_0 := \mathbb{P}(\,\omega \,:\, \varphi(t_0,\omega)x, \varphi(t_0,\omega)y \in B_r(p)\,) \;>\; 0.$$

[8] cf. part (b) of the proof of [14, Proposition 2], or [7, Lemma 2.19(2)].

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Then we have that

$$\begin{aligned}&\mathbb{P}(\omega : x \sim_\omega y)\\ &\quad\geq\ \mathbb{P}(\,\omega\ :\ \varphi(t_0,\omega)x, \varphi(t_0,\omega)y \in B_r(p)\ \text{ and }\ \varphi(t_1,\theta^{t_0}\omega)\overline{B_r(p)} \subset \bar{U}\ \text{ and }\ \theta^{t_0+t_1}\omega \in E_V\,)\\ &\quad=\ k_0 k_1 \mathbb{P}(E_V)\\ &\quad>\ 0\end{aligned}$$

as required. So in particular, $(A_1 \times A_2) \setminus \Delta_X \ \subset\ Y$. Now since $1 = \rho(A_2) = \int_\Omega \mu_\omega(A_2)\,\mathbb{P}(d\omega)$, we have that $\mu_\omega(A_2) = 1$ for $\mathbb{P}$-almost all $\omega \in \Omega$, and therefore

$$\bar{\mu}^{(2)}(A_1 \times A_2) \ =\ \int_\Omega \mu_\omega(A_1)\mu_\omega(A_2)\,\mathbb{P}(d\omega) \ =\ \int_\Omega \mu_\omega(A_1)\,\mathbb{P}(d\omega) \ =\ \rho(A_1).$$

Let n_ρ be as in Proposition 2.5 (meaning in particular that (μ_ω) is n_ρ-uniform, as in the proof of Proposition 2.5 outlined further above). By Remark 3.1, we have that

$$\bar{\mu}^{(2)}((A_1 \times A_2) \cap \Delta_X) \ =\ \bar{\mu}^{(2)}(\Delta_{A_1 \cap A_2}) \ =\ \tfrac{1}{n_\rho}\rho(A_1),$$

and therefore

$$\bar{\mu}^{(2)}\left((A_1 \times A_2) \setminus \Delta_X\right) \ =\ \tfrac{n_\rho - 1}{n_\rho}\rho(A_1).$$

But since $(A_1 \times A_2) \setminus \Delta_X \ \subset\ Y$, we also have that

$$\bar{\mu}^{(2)}\left((A_1 \times A_2) \setminus \Delta_X\right) \ =\ 0.$$

Since $\rho(A_1) \neq 0$, it obviously follows that $n_\rho = 1$, i.e. (iii) holds.

Now suppose that (iii) holds; we show that (iv) holds. As in Remark 2.4, let $A \subset X$ be a ρ-full set such that for all $x, y \in A$, $\mathbb{P}(\omega : x \sim_\omega y) = 1$; without loss of generality, take A to be a subset of $\operatorname{supp}\rho$ such that for each $x \in A$, $\mathbb{P}$-almost every $\omega \in \Omega$ has the property that for all $T \in \mathbb{T}^+$, $\{\varphi(t,\omega)x : t \geq T\}$ is dense in $\operatorname{supp}\rho$. Fix any $x, y \in A$, and let ω be any sample point with the properties that $x \sim_\omega y$ and for all $T \in \mathbb{T}^+$, $\{\varphi(t,\omega)x : t \geq T\}$ is dense in $\operatorname{supp}\rho$. Fix any open $U \subset X$ with $\rho(U) > 0$, and let $p \in U$ and $\varepsilon > 0$ be such that $B_\varepsilon(p) \subset U$. Let $T \in \mathbb{T}^+$ be such that for all $t \geq T$, $d(\varphi(t,\omega)x, \varphi(t,\omega)y) < \frac{\varepsilon}{2}$; and let $t' \geq T$ be such that $\varphi(t',\omega)x \in B_{\frac{\varepsilon}{2}}(p)$. Then both $\varphi(t',\omega)x$ and $\varphi(t',\omega)y$ are in $B_\varepsilon(p)$ and hence in U.

Finally, it is clear that (iv)$\Rightarrow$(i) (with $R = \operatorname{supp}\rho$). $\square$

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